Topological gauge/string theory and enumeration

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1 Introduction

This note gives a pedagogical introduction to topological string theory. Recent interest in topological string is largely due to the OSV conjecture $Z_{BH} = |Z_{top}|^2$, which relates the partition function Z_{BH} (for a mixed canonical ensemble)of the extremal four dimensional black holes and the partition function Z_{top} of topological string. In this talk I will focus on the following two Physical Aspects of Topological String Theory from the viewpoint of the OSV conjecture.

- Topological string counts "stable" objects (BPS states, instantons, solitons, $\cdots$). The counting becomes often easier in a (holographic) dual CFT picture, or IR limit of a dual gauge theory on branes. This is regarded as a topological version of gauge/string correspondence. For example, GW/DT correspondence is proposed in mathematics.

- Topological string computes a certain F-term (holomorphic piece) in the low energy effective action of $\mathcal{N} = 2$ supergravity, including R^2 correction terms. This fact has been crucial in most of recent physical application of topological string theory.

To explain the reason why I think these aspects are significant, let me present a few key words in the computation of the entropy of extremal (4D) black holes from string theory.

- Macroscopic Side
  This is the side where OSV conjecture was originally derived. Special geometry of $\mathcal{N} = 2$ (4D) supergravity combined with the attracter mechanism for extremal (BPS) black holes is heavily used. By the special geometry relations the central charges of $\mathcal{N} = 2$ SUSY is expressed in terms of the prepotential which appears in the low energy effective action describing the coupling of (abelian) gauge multiplets and gravity multiplet. In type IIB compactification the prepotential is related to the geometry of Calabi-Yau manifold through the period integral (*cf.* Seiberg-Witen theory). Furthermore, the attracter mechanism for extremal black hole tells us that the near horizon geometry and the horizon area depend only on their electric-magnetic charges. There are higher derivative corrections to the entropy formula coming from R^2 term in low energy effective action of $\mathcal{N} = 2$ (4D) supergravity.

- Microscopic Side
  Several attempts have been made on this side, to confirm or establish the OSV conjecture. We count the degeneracy of BPS states as bound states of solitons in string theory, or D-brane configurations. In a dual holographic CFT descriptions, which is IR limit of the worldvolume theory on D-branes, (for example, SUSY sigma model whose target space is the symmetric product of $K3$ surface; a chiral $(0, 4)$ CFT arising from $M5$ branes), the counting is achieved by computing a generalized supersymmetric index (sometimes called elliptic genus). The index is independent of the coupling constant and we may compute it in the region where solitons are weakly coupled. A gravitational (geometric) picture emerges in the strong coupling side and the solitons are identified as extremal black holes.

2 Formulation

Generalities on topological quantum field theory

Dynamical informations of quantum field theory are obtained by computating the partition function and the correlation function (more generally a generating function of the correlation functions or low energy effective action). If the theory is defined by the (microscopic) action $S[\phi] = \int_M d^n x \, \mathcal{L}[\phi]$ for a dynamical field variable $\phi(x)$ on the space-time M, the partition function is formally defined by the (Euclidean) path integral;

$$Z := \int [\mathcal{D}\phi] e^{-S[\phi]} \ , \tag{1}$$

and the N-point function (vacuum expectation values) is

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_N \rangle := \frac{1}{Z} \int [\mathcal{D}\phi] \mathcal{O}_1[\phi] \mathcal{O}_2[\phi] \cdots \mathcal{O}_N[\phi] e^{-S[\phi]} \ , \tag{2}$$

where $\mathcal{O}_i[\phi]$ are "observables" or physical operators of the theory. Note that the choice of physical operators (together with the vacuum $|0\rangle$) is a part of the definition of quantum field theory.

In general the action $S[\phi]$ and the operators $\mathcal{O}_i$ depends on various parameters such as (a background) metric $g_{\mu\nu}$ on M, coupling constants and the positions of $\mathcal{O}_i$. Hence the partition function Z and the correlation functions $\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_N \rangle$ also depend on these parameters. In particular, if they are invariant (or rigid) under continuous local deformations of background metric,

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_N \rangle = 0, \tag{3}$$

they are called topological partition function or topological correlation functions. This condition physically implies there is no local dynamics, or no propagating modes in the theory. Since this means the correlation function is invariant under local translation (diffeomorphism) $\delta x = \epsilon(x)$, we expect that topological correlation function is independent of the positions of the operator;

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_N \rangle \sim \text{constant} \tag{4}$$

(There are possibilities of the so-called contact term which appears when more than two operators collide or intersect.) These constants are mathematically identified as topological "index" obtained by "counting" appropriate objects, intersection number, linking number *etc.* or the (phase space) volume of the moduli space, the parameter space of the classical equation of motion. Physically it is interpreted as the number of (stable) states such as solitons and BPS states. (Note that in quantum theory the phase space volume in the unit of $\hbar$ gives the number of quantum states.) If all the correlation functions in the theory (note that this depends on a choice of vacuum and physical opearators, which might be empty; $Z = \langle 0|0\rangle = 1$ is the only "topological" quantities.), it is called called topological quantum field theory. Or by appropriate choice of vacuum together with a set of physical operators in a given theory, one might define a topological sector of the original theory.

When can we have topological quantum field theory? Such an exotic theory may occur, when

$$\frac{\delta S}{\delta g_{\mu\nu}} = T_{\mu\nu} = 0 \ , \tag{5}$$

which is the case when either the classical action is a topological invariant or the action does not involve any background metric. Formal manipulation (integration by parts) in path integral implies

$$\begin{aligned} \frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_N \rangle &\sim \int [\mathcal{D}\phi] \frac{\delta}{\delta g_{\mu\nu}} (\mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_N) e^{-S} \\ &= \int [\mathcal{D}\phi] \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_N \frac{\delta}{\delta g_{\mu\nu}} (e^{-S}) = 0 \ . \end{aligned} \tag{6}$$

Typical examples are two dimensional gravity and the Chern-Simons gauge theory in three dimensions. The Einstein-Hilbert action

$$S_{\rm EH}[g] := \frac{1}{16\pi G_N} \int d^N x \sqrt{g} \ R[g] \tag{7}$$

is topological (it just gives the Euler number) in two ($N = 2$) dimensions. The Newton constant G_N is dimensionless only in two dimensions. We have already seen the Chern-Simons action in several talks in this SI. For examples, the fact that three dimensional

CS action does not have any dynamical degrees of freedom plays an important role in the construction of BLG action. The action of the Chern-Simons theory

$$S_{\rm CS}[A] := \frac{k}{4\pi}\int_{M^3} {\rm Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{8}$$

contains no metric and topological.

The idea of topological quantum field theory was first introduced by Witten in Feb. 1988 (twenty years ago) from a rather mathematical motivation of describing the Donaldson invariants as correlation functions of quantum field theory. A physical interpretation he suggested is that TQFT describes a topological phase (in general relativity) where general covariance is unbroken and massless graviton is confined. Note that the introduction of a (background) metric necessarily breaks the general covariance and massless gravitons are interpreted as Goldstone bosons arising from the spontaneous breaking of the general covariance. If we could perform the path integral over all the possible metric in quantum gravity without introducing a classical metric such as a flat metric, we might obtain an effective theory with general covariance. It is interesting that topological quantum field theory achieves general covariance without integrating out the background metric. It seems that this is very similar to our approach to the low energy effective action, where we consider what one would obtain after integrating out (irrelevant) massive degrees of freedom, even if we cannot make the path integral explicitly. In fact Witten proposed that TQFT might be obtained as low energy effective theory to some kind of string field theory in a phase where general covariance is unbroken. As an example, he showed that the Chern-Simons theory on M^3 is obtained as a low energy effective theory of open string field theory on the cotangent bundle T^*M^3.

To illustrate these abstract discussions on topological theories, let us take the Nambu-Goto action of string theory;

$$S_{\rm NG}[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma d\tau d\sigma\sqrt{-\det\, h}\ , \quad X^\mu = X^\mu(\tau,\sigma), \quad T := -\frac{1}{2\pi\alpha'}\ , \tag{9}$$

where $h_{ab} := \partial_a X^\mu \partial_b X_\mu$ is an induced metric. Since this is proportional to the area of the string world-sheet, it is "topological" and has general covariance. The action is invariant under the diffeomorphism of the string world-sheet. Note that the Numbu-Goto action does not employ any metric on Σ, and thus it gives an example of general covariant theory without metric. However, it is difficult to quantize the Numbu-Goto action. Introducing (an auxiliary) world-sheet metric γ_{ab} we can write down the following action which is suitable for quantization;

$$S_{\rm P}[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \ . \tag{10}$$

Classical equivalence of $S_{\rm NG}$ and $S_{\rm P}$ is seen as follows; the equation of motion for the auxiliary metric γ_{ab} is

$$\frac{1}{2}\gamma_{ab}(\gamma_{cd}\partial^c X^\mu \partial^d X_\mu) = \partial_a X^\mu \partial_b X_\mu \ . \tag{11}$$

Thus the metric γ_{ab} is conformal transform of $\partial_a X^\mu \partial_b X_\mu$ with conformal weight $\rho = 2/(\gamma_{cd}\partial^c X^\mu \partial^d X_\mu)$. Substituting the above relation to $S_{\rm P}[X,\gamma]$, we find the conformal weight cancels and $S_{\rm P}[X,\gamma]$ reduces to the Nambu-Goto action. In quantum theory we expect we can obtain a theory with general covariance by integrating out the auxiliary metric γ_{ab}.

In quantum theory the variation of the action with respect to the metric gives first class constraints on the energy-momentum tensor;

$$T_{ab} := \frac{\delta S_{\rm P}}{\delta \gamma^{ab}} \simeq 0 \tag{12}$$

which is regarded as a consequence of the general covariance. Due to the existence of the first class constraints, we have to fix a gauge. A modern way of fixing a gauge in quantum theory is the BRST quantization. In the BRST formalism the first class constraints are promoted into

$$T_{ab} = \{Q_B, b_{ab}\} \ , \tag{13}$$

where Q_B is a generator of fermionic symmetry which is scalar and satisfies $Q_B^2 = 0$. The last condition allows us to introduce the BRST cohomology. The above relation means the energy momentum is trivial in the sense of cohomology. We can say that this is a characterizing property of topological quantum field theory. When the theory is topological in the sense of BRST cohomology, it is sometimes called "cohomological" theory. The theory has a scalar fermionic symmetry Q_B which satisfies the relation of the coboundary operator $Q_B^2 = 0$.

It is crucial that we can obtain such a special symmetry $Q_B^2 = 0$ and $T_{\mu\nu} = \{Q_B, \lambda_{\mu\nu}\}$ by twisting the extended supersymmetry algebra $T = \{G^+, G^-\}$. The generators of the supersymmetry transform as spinors. The twisting is a redefinition of the Lorentz generators or the spin connections and some of the supercharges become scalar under the redefined Lorentz generators.

Topological twist of $\mathcal{N} = (2,2)$ SUSY sigma-model

Usually string theory is perturbatively defined in terms of the genus expansion. Namely we consider the string world sheet Σ_g of a fixed genus g and string amplitudes are computed genus by genus. When the string moves on a (Calabi-Yau) manifold M, we consider a map

$$\phi : \Sigma_g \to M \ , \tag{14}$$

and the corresponding quantum field theory is called (two-dimensional) sigma model on Σ_g with the target space M. The genus g string amplitudes for process on the space-time M are computed by integrating correlation functions on the world sheet over the moduli space of Σ_g, which means physically the coupling to the gravity on Σ_g.

When the target manifold is a Kähler manifold, we can introduce $\mathcal{N} = (2,2)$ supersymmetry on the world sheet and the theory on the world sheet is called two dimensional $\mathcal{N} = (2,2)$ supersymmetric sigma model. To describe it, we introduce the superfields Φ^I and $\Phi^{\bar{I}}$ whose lowest components are the bosons

$$\phi^{I,\bar{I}} : \Sigma_g \to M \ . \tag{15}$$

Its super partners are two Dirac spinors

$$\psi^I_{\pm,+} \in \Gamma(\Sigma_g, S^{\pm} \otimes \phi^*(T^{(1,0)}M)) \ , \qquad \psi^{\bar{I}}_{\pm,-} \in \Gamma(\Sigma_g, S^{\pm} \otimes \phi^*(T^{(0,1)}M)) \ , \tag{16}$$

where $S^{\pm}$ are the positive and the negative chirality spinor bundles on Σ_g with spins $\pm\frac{1}{2}$ w.r.t two dimensional rotation group $U(1)_E$. The complexified tangent bundle $TM \otimes \mathbb{C}$ is decomposed into the holomorphic and the anti-holomorphic tangent bundles

$$TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M \ , \tag{17}$$

and $\phi^*(T^{(1,0)}X)$ and $\phi^*(T^{(0,1)}X)$ are their pull-backs to Σ_g. (The left indices $\pm$ stand for the chirality on the world sheet Σ_g and the right indices $\pm$ specify the holomorphic and the anti-holomorphic coordinates on the target space.) The superfields Φ^I and $\Phi^{\bar{I}}$ also contain auxiliary fields F^I and $F^{\bar{I}}$.

Let us look at the kinetic terms of fermions, since the twist operator only affects this part;

$$S_{f,\mathrm{kin}} = \int_{\Sigma_g} d^2z \ G_{I\bar{J}}(\phi) \left[\psi^{\bar{J}}_{+,-} D_{\bar{z}} \psi^I_{+,+} + \psi^{\bar{J}}_{-,-} D_z \psi^I_{-,+}\right] \ , \tag{18}$$

where the covariant derivatives are defined by

$$\begin{aligned} D_{\bar{z}}\psi^I_{+,+} &= \partial_{\bar{z}}\psi^I_{+,+} + \frac{i}{2}\omega_{\bar{z}}\psi^I_{+,+} + \Gamma^I_{KL}(\phi)\partial_{\bar{z}}\phi^K\psi^L_{+,+} \ , \\ D_z\psi^I_{-,+} &= \partial_z\psi^I_{-,+} - \frac{i}{2}\omega_z\psi^I_{-,+} + \Gamma^I_{KL}(\phi)\partial_z\phi^K\psi^L_{-,+} \ . \end{aligned} \tag{19}$$

They have the spin connection $\omega_{z,\bar{z}}$ of $U(1)_E$ on Σ_g and the pull-backs $\Gamma^I_{KL}\partial_{z,\bar{z}}\phi^K$ of the (Hermitian) connection Γ^I_{KL} of the Kähler metric. The model has a global abelian symmetry $U(1)_V \times U(1)_A$, which is called R symmetry. The associated currents are a conserved vector current

$$j^z_V = G_{I\bar{J}}\psi^{\bar{J}}_{-,-}\psi^I_{-,+} \ , \qquad j^{\bar{z}}_V = G_{I\bar{J}}\psi^{\bar{J}}_{+,-}\psi^I_{+,+} \ , \tag{20}$$

and an anomalous axial vector current

$$j^z_A = -G_{I\bar{J}}\psi^{\bar{J}}_{-,-}\psi^I_{-,+}(= -j^z_V) \ , \qquad j^{\bar{z}}_A = G_{I\bar{J}}\psi^{\bar{J}}_{+,-}\psi^I_{+,+}(= j^{\bar{z}}_V) \ . \tag{21}$$

The anomaly of j_A is given by the index of Dirac operator

$$\partial_\mu j^\mu_A = \int_{\Sigma_g} \phi^*(c_1(M)) \ , \tag{22}$$

where $c_1(M)$ is the 1st Chern class of M.

The topological twist of the model may be introduced by promoting a $U(1)$ subgroup of the global R-symmetry $U(1)_V \times U(1)_A$ to a local symmetry (and identify the corresponding gauge connection with the $U(1)_E$ spin connection). That is the action of the twisted model is obtained by adding to the original action the coupling of the corresponding currents j_V and j_A with the spin connection. The twisting by j_V gives the A model

$$\begin{aligned} & S_{\mathrm{f}} - \frac{i}{2}\int_{\Sigma_g} d^2z \ \omega_\mu \ j_V^\mu \\ = & \int_{\Sigma_g} d^2z \ G_{I\bar{J}} \left[\psi^{\bar{J}}_{+,-} \left(\partial_{\bar{z}} + \frac{i}{2}\omega_{\bar{z}} \right) \psi^I_{+,+} + \psi^{\bar{J}}_{+,-} \Gamma^I_{KL} \partial_{\bar{z}} \phi^K \psi^L_{+,+} \right. \\ & \left. + \psi^{\bar{J}}_{-,-} \left(\partial_z - \frac{i}{2}\omega_z \right) \psi^I_{-,+} + \psi^{\bar{J}}_{-,-} \Gamma^I_{KL} \partial_z \phi^K \psi^L_{-,+} - \frac{i}{2}\left(\omega_z \ \psi^{\bar{J}}_{-,-} \psi^I_{-,+} + \omega_{\bar{z}} \ \psi^{\bar{J}}_{+,-} \psi^I_{+,+} \right) \right] \\ = & \int_{\Sigma_g} d^2z \ G_{I\bar{J}} \left[\psi^{\bar{J}}_{+,-} \partial_{\bar{z}} \psi^I_{+,+} + \psi^{\bar{J}}_{+,-} \Gamma^I_{KL} \partial_{\bar{z}} \phi^K \psi^L_{+,+} + \psi^{\bar{J}}_{-,-} \left(\partial_{\bar{z}} - i\omega_{\bar{z}} \right) \psi^I_{-,+} + \psi^{\bar{J}}_{-,-} \Gamma^I_{KL} \partial_z \phi^K \psi^L_{-,+} \right] \end{aligned} \tag{23}$$

By looking at the coupling to the spin connection, we see that the spin of the fermions is changed as follows;

$$\begin{array}{ll} \psi^I_{+,+} : +\frac{1}{2} \rightarrow 0 \ = \text{scalar} & \psi^I_{-,+} : -\frac{1}{2} \rightarrow -1 \ = (0,1) \text{ form} \\ \psi^{\bar{J}}_{+,-} : +\frac{1}{2} \rightarrow +1 \ = (1,0) \text{ form} & \psi^{\bar{J}}_{-,-} : -\frac{1}{2} \rightarrow 0 \ = \text{scalar} \ . \end{array} \tag{24}$$

Similarly the twisting by j_A gives the B model.

According to the change of spin of fermions, we introduce the following notation

$$\chi^I = \psi^I_{+,+} \ , \quad \chi^{\bar{J}} = \psi^{\bar{J}}_{-,-} \ , \qquad \rho^I_{\bar{z}} = \psi^I_{-,+} \ , \quad \rho^{\bar{J}}_z = \psi^{\bar{J}}_{+,-} \ , \tag{25}$$

$\chi \in \Gamma(\Sigma_g, \phi^*(T^{(1,0)}M \oplus T^{(0,1)}M))$ is a scalar on Σ_g and $\rho_\alpha \in \Gamma(\Sigma_g, K \otimes \phi^*(T^{(0,1)}M) \oplus \overline{K} \otimes \phi^*(T^{(1,0)}M))$ is a one form on Σ_g. (The canonical line bundle K is the bundle of $(1,0)$ forms on Σ_g and the anti-canonical bundle $\overline{K}$ is the bundle of $(0,1)$ forms on Σ_g.)

The full action after the twist of type A is

$$S_A = \int_{\Sigma_g} d^2z \left[G_{I\bar{J}} \left(g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^{\bar{J}} + \frac{i\epsilon^{\mu\nu}}{\sqrt{g}}\partial_\mu\phi^I\partial_\nu\phi^{\bar{J}} - g^{\mu\nu}\rho^I_\mu D_\nu\chi^{\bar{J}} - g^{\mu\nu}\rho^{\bar{J}}_\mu D_\nu\chi^I - \frac{1}{2}g^{\mu\nu}\tilde{F}^I_\mu\tilde{F}^{\bar{I}}_\nu \right) + \frac{1}{2}g^{\mu\nu}R_{\bar{I}J\bar{K}L}\rho^{\bar{I}}_\mu\rho^J_\nu\chi^{\bar{K}}\chi^L \right] . \tag{26}$$

We have redefined the original auxiliary fields $F^I_z, F^{\bar{I}}_{\bar{z}}$ to $\tilde{F}^I_z, \tilde{F}^{\bar{I}}_{\bar{z}}$. $\mathcal{N}=(2,2)$ SUSY sigma model has four fermionic charges $Q_{\pm,\pm}$. After the A twist $Q_{+,+}$ and $Q_{-,-}$ are scalar and $Q_{+,-}$ and $Q_{-,+}$ have spin ± 1. We define

$$Q = Q_{+,+} + Q_{-,-} , \quad G_z = Q_{+,-} , \quad G_{\bar{z}} = Q_{-,+} . \tag{27}$$

Then the $\mathcal{N}=(2,2)$ SUSY algebra implies

$$Q^2 = 0 , \qquad \{Q, G_\alpha\} = H_\alpha . \tag{28}$$

We will identify the nilpotent charge Q as a BRST operator of the A model. Then we have

$$S_A = \{Q, V\} + i\int_{\Sigma_g} \phi^*(\omega) \tag{29}$$

where

$$\int_{\Sigma_g} \phi^*(\omega) = \int_{\Sigma_g} d^2z G_{I\bar{J}} \left(\partial_z\phi^I\partial_{\bar{z}}\phi^{\bar{J}} - \partial_{\bar{z}}\phi^I\partial_z\phi^{\bar{J}} \right) \tag{30}$$

is the pull-back of the Kähler form $\omega = G_{I\bar{J}}dx^I \wedge dx^{\bar{J}}$ of the target and

$$V = \frac{1}{2}\int_{\Sigma_g} d^2z\sqrt{g}g^{\mu\nu}G_{I\bar{J}} \left[\frac{1}{2}\rho^I_\mu\tilde{F}^{\bar{J}}_\nu + \frac{1}{2}\rho^{\bar{J}}_\mu\tilde{F}^I_\nu + \left(\rho^I_\mu\partial_\nu\phi^{\bar{J}} + \rho^{\bar{J}}_\mu\partial_\nu\phi^I \right) \right] . \tag{31}$$

Thus the action of topological A model is BRST exact modulo the topological term $\int_{\Sigma_g} \phi^*(\omega)$ which counts the winding number (or the instanton number) of the map $\phi : \Sigma_g \to M$. We can add a coupling to a B-field (= a background of 2-form flux on M) $\int_{\Sigma_g} \phi^*(B)$. Then we have a complexified Kähler form $J = B + i\omega$.

Since we have

$$T_{\mu\nu} = \frac{\delta S_A}{\delta g^{\mu\nu}} = \{Q, \frac{\delta V}{\delta g^{\mu\nu}}\} = \{Q, G_{\mu\nu}\} , \tag{32}$$

topological A-model is two dimensional topological theory with Q as BRST operator. The observable of the model are defined to be Q-cohomology class. Namely observables $\mathcal{O}$ satisfies $[Q, \mathcal{O}]_\pm = 0$ and $\mathcal{O} \sim \mathcal{O}'$ if $\mathcal{O} = \mathcal{O}' + [Q, \Lambda]_\pm$. If the vacuum is Q invariant; $Q|0\rangle$, the correlation function $\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle := \langle 0 | \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n | 0 \rangle$ depends only on the the cohomology class of the observables

$$\mathcal{O} = \mathcal{O}' + [Q, \Lambda]_\pm \Longrightarrow \langle \mathcal{O} \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \langle \mathcal{O}' \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \ , \tag{33}$$

and is topological invariant in the following sense

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle = \langle T_{\mu\nu} \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = 0 \ . \tag{34}$$

It terns out that observable in topological A model is in one to one correspondence to the de Rham cohomology of the target space.

Topological string = Coupling to (topological) gravity

Topological string theory is defined by coupling topological sigma model with topological gravity on the string world sheet. This coupling can be achieved in the same way to the (physical) bosonic string theory, by making use of the similarities of BRST structures of both theory. In the path integral formulation the coupling to $2D$ gravity is defined by an integral over the moduli space $\mathcal{M}_g$ of Riemann surface of genus g. We note that the same integral is used in the perturbative formulation of *bosonic* string theory, where the energy momentum tensor is given by

$$T(z) = \{Q_B, b(z)\} \ , \tag{35}$$

The point is that the twisted topological sigma model has the same BRST structure[1];

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\} \ . \tag{36}$$

[1]Here $T_{\mu\nu}$ and $G_{\mu\nu}$ are Noether currents. Compare with (28) where the algebra is written in terms of charges.

Thus we can completely follow the prescription of the perturbative definition of the amplitudes of bosonic string theory and define *perturbative* topological string amplitudes. The free energy of genus $g > 1$ is

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{k=1}^{6g-6} (G, \mu_k) \rangle \ ,$$
$$(G, \mu_k) = \int_{\Sigma_g} (G_{zz} (\mu_k)^z_{\bar{z}} + G_{\bar{z}\bar{z}} (\bar{\mu}_k)^{\bar{z}}_z) \ . \tag{37}$$

The case of $g = 0, 1$ requires special cares, since there are symmetries (Killing vectors) in the lower genus surface. On the Riemann surface of general type $g \geq 1$, there are $6g - 6$ Beltrami differential $\mu_k = (\mu_k)^z_{\bar{z}} d\bar{z} \otimes \frac{\partial}{\partial z} \in H^1_{\bar{\partial}}(\Sigma_g, T^{(1,0)}\Sigma_g)$. They are associated with the deformation of complex structure of Σ_g, whose moduli space $\mathcal{M}_g$ has dimensions $6g - 6$. The integrand gives $6g - 6$ form on $\mathcal{M}_g$.

The topological correlation function $\langle \prod_{k=1}^{6g-6} (G, \mu_k) \rangle$ can be computed by semi-classical approximation of path integral, which reduces to the integral over the moduli space, the parameter space of equation of motion. Or the path integral localizes to the fixed points of BRST transformations. From the BRST transformation law of the anti-ghost ρ, which gives EOM (gauge fixing condition) of topological theory, we see that the (world sheet) instantons are holomorphic maps $\phi : \Sigma \to M$. The holomorphic maps are topologically classified by instanton number (or the winding number) as follows; Let us introduce a basis $[S_i], i = 1, \cdots, b_2(M) = \dim H_2(M, \mathbb{Z})$ of the second homology of M. Then the topological types of instantons is classified by the (second) homology class $\beta = \phi_*[\Sigma_g] \in H_2(M, \mathbb{Z}) = \sum_{i=1}^{b_2(M)} n_i [S_i]$. The integers n_i are called instanton number. The weight of the instanton sector comes from the topological term in the A model action.

$$S_A = \{Q, V\} + \int_{\Sigma_g} \phi^*(J) \tag{38}$$

Recall that $J = B + i\omega$ is the complexified Kähler form. In terms of the (complexified) Kähler parameter $t_i = \int_{S_i} J$ each sector is weighted with $Q^\beta = \prod_{i=1}^{b_2(M)} Q_i^{n_i}$, with $Q_i = e^{-t_i}$. Thus we obtain the following form of the free energies F_g, $(g \geq 1)$ of the type A

topological string;

$$F_g(t) = \sum_{\beta \in H_2(M,\mathbb{Z})} N_{g,\beta} Q^\beta \ . \tag{39}$$

The number $N_{g,\beta}$ is called the Gromov-Witten invariants of M at genus g in the class β. Finally the partition function of (perturbative) topological string is defined by

$$Z_{top} = \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} F_g(t)\right) \ , \tag{40}$$

where the string coupling g_s plays the role of parameter of genus expansion. Since the topological term depends only on the (complexified) Kähler form $J = B + i\omega$, topological partition function (which also gives a generating function of topological correlation function of two form observables) are functions of the Kähler moduli $\{t^i\}, i = 1, \cdots b_2(M)$.

3 Computation

BPS state counting in five dimensions

Let us consider a compactification of M theory to five dimensions on a Calabi-Yau threefold X. M theory on X has $U(1)^n$ gauge symmetry where $n = b_2(X)$, since a dimensional reduction of M theory three-form C on each two-cycle gives rise to a $U(1)$ gauge field. An $M2$ brane wrapped on a two-cycle gives a charged particles under the corresponding $U(1)$. We could ask the number of BPS states with a given charge $Q \in H_2(X,\mathbb{Z})$. Actually in five dimensions BPS states can carry non-vanishing spin. In five dimensions the spin of massive particles is classified by $SO(4) \simeq SU(2)_L \times SU(2)_R$ and BPS states have to satisfy $j_L = 0$ or $j_R = 0$. Hence we want to count the number of BPS states with charge Q and spin j_L.

We can find these BPS state degeneracies, which is called the Gopakumar-Vafa invariants from topological string amplitude. This is closely related to the fact that topological string amplitudes at genus g computes the following F-term in the low energy effective action of the four dimensional $\mathcal{N}=2$ SUSY theory obtained by type IIA string on X;

$$\int d^4x \int d^4\theta F_g(t)(\mathcal{W}^2)^g + \text{c.c.} = \int d^4x F_g(t)\left(R_+^2 F_+^{2g-2} + R_-^2 F_-^{2g-2}\right) \ . \qquad (41)$$

The reasoning roughly goes as follows; in type IIA compactification the coefficients $F_g(t)$ is obtained by computing the (super)string amplitude of 2 gravitons and $2g-2$ graviphotons. Since the graviphoton comes from RR sector, looking at the couping of RR states and the dilaton, which is in the hypermultiplets in type II theory, we see that $F_g(t)$ is exactly given by genus g amplitude. Another crucial point is that the insertion of $(2g-2)$ graviphotons induces an effective coupling $\int_{\Sigma_g} \frac{\sqrt{3}}{2} H \cdot R^{(2)}$, where $\partial H = j_V$ is the $U(1)$ current which we have used to twist the SUSY sigma model. Finally after the cancellation of contributions from bosons and fermions, we end up with the integration over the supermoduli space of the worldsheet Σ_g. Integration of the Grassmannian coordinates of the supermoduli and the cancellation of superghost charge can be made by inserting an appropriate number (namely $3g-3$) of the picture changing operator, which, in this case, is given by the supercurrents $G^{\pm}$. Thus we find the topological string amplitude we have defined before.

Thus if we consider a self-dual graviphoton background $F_+ = g_s, F_- = 0$ in the Euclidean version of the theory, the coupling in the effective action to R_+^2 is nothing but the all genus topological string amplitude

$$\sum_{g=0}^{\infty} F_g(t) g_s^{2g-2} \qquad (42)$$

On the otherhand, we can obtain the same R_+^2 correction terms from the effective action of charged particle, and we can compute it exactly if graviphoton background is constant, as we will see shortly. In the strong coupling limit only the BPS multiplets contribute to the low energy effective action. This gives a basic identification of (A model) topological string amplitude with a generalized SUSY index which gives the degeneracy of BPS states. In the computation of the R_+^2 correction to the low energy effective action, the relevant charged particles are the quanta of $\mathcal{N}=2$ hypermultiplets obtained by quantizing the wrapped $D2$ and $D0$ branes. If $D2$-brane is wrapped on a cycle Q and bound to k $D0$-branes, the central charge is $Z = \langle Q, t\rangle + ik$ and the BPS state has the mass $m = |Z|$. Let us consider the non-perturbative correction to the low energy effective action due to the pair production of such BPS particles. Since they are in the hypermultiplet the contribution to the vacuum energy cancels, but there is non-trivial contribution to R_+^2 term. It turns out that it is precisely the same as the lowest component of the multiplet would contribute to the vacuum energy, which can be computed non-perturbatively by the proper time or heat kernel expansion as follows;

Let us consider a charged particle of charge q and mass m in a constant self-dual $U(1)$ flux in four dimensional (Eucilidian) space. We will compute the contribution to (non-perturbative) low energy effective action, from the charged (BPS) particle in a constant gravi-photon background [2]

$$F_{12} = F_{34} = g_s\mu^2 \ . \tag{43}$$

The low energy effective action is obtained by the path integral of the microscopic action

$$S = |(\partial_i - iqA_i)\phi|^2 + m^2|\phi|^2 \ , \tag{44}$$

for complex scalar ϕ, which is Gaussian. We obtain

$$S_{\rm eff} = \log\det\,(\Delta_{12} + \Delta_{34} + m^2) \ , \tag{45}$$

where $\Delta_{ij} = D_i^2 + D_j^2$ is the covariant Laplacian with gauge potential. Note that

$$[D_1, D_2] = [D_3, D_4] = iqg_s\mu^2 \ . \tag{46}$$

Using the heat kernal expansion (the proper time expansion), we have

$$\log\det\,(\Delta_{12} + \Delta_{34} + m^2) = {\rm Tr}\log(\Delta_{12} + \Delta_{34} + m^2) = \int_\epsilon^\infty \frac{dt}{t}{\rm Tr}\ e^{-t(\Delta_{12}+\Delta_{34}+m^2)} \ . \tag{47}$$

From the computation of the partition function of the harmonic oscillator, we find

$${\rm Tr} e^{-t(\Delta_{12}+\Delta_{34}+m^2)} = e^{-tm^2}\left(\sum_{n=0}^\infty e^{-tqg_s\mu^2(n+\frac{1}{2})}\right)^2 = \frac{e^{-tm^2}}{\left(2\sinh\frac{tqg_s\mu^2}{2}\right)^2} \ , \tag{48}$$

[2]Note that g_s should be dimensionless. We have inserted μ to keep track of the mass dimension.

Thus we finally obtain

$$S_{\text{eff}} = \int_{\epsilon}^{\infty} \frac{d\tilde{t}}{\tilde{t}} \frac{e^{-q\tilde{t}}}{\left(2\sin\frac{\tilde{t}g_s}{2}\right)^2} , \tag{49}$$

where we have used the BPS condition $m = q\mu$ and put $\tilde{t} = tq\mu^2$. By an appropriate regularization this agrees to the free energy of $c = 1$ string theory at self-dual radius;

$$F_{c=1}(m;\Lambda) = g_s^{-2}\left(\frac{1}{2}m^2\log\frac{m}{\Lambda} - \frac{3}{4}m^2\right) - \frac{1}{12}\log\frac{m}{\Lambda} + \sum_{g=2}^{\infty} g_s^{2g-2}\frac{B_{2g}}{2g(2g-2)}m^{2-2g} , \tag{50}$$

where Λ is a regularization parameter and B_{2g} is the Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n , \quad B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}\cdots , \quad (B_{2k+1} = 0, k > 0) \tag{51}$$

In $c = 1$ string theory m is identified with the cosmological constant. It is remarkable that the coefficients of this genus expansion gives the Euler number of the moduli space $\mathcal{M}_g$ of the Riemann surface with gunes g. This is somewhat a surprising result, one of the miracles in string theory. We have just computed an effective action of charged BPS particle in a constant flux. There is no Riemann surface at all. But the expansion coefficients in g_s automatically gives informations on the moduli space of Riemann surface. Thus the BPS particle somehow knows the geometry of string worldshhet.

Now the charge spectrum of $D2$-$D0$ system is

$$Z = \langle q, t\rangle + 2\pi i k, \qquad k \in \mathbb{Z}\setminus\{0\} . \tag{52}$$

Summing up the number of $D0$ branes, or the K-K modes of S^1 compactification, we obtain

$$F(g_s) = \sum_{k\in\mathbb{Z},k\neq 0}\int_{\epsilon}^{\infty}\frac{ds}{s}\frac{e^{-2\pi iks-\langle q,t\rangle s}}{\left(2\sinh\frac{sg_s}{2}\right)^2} . \tag{53}$$

By the Poisson resummation formula

$$\sum_{k\in\mathbb{Z},k\neq 0} e^{2\pi iks} = \sum_{m\in\mathbb{Z}}\delta(s-m) , \tag{54}$$

we obtain

$$F(g_s) = \sum_{m=1}^{\infty}\frac{1}{m}\frac{e^{-m\langle q,t\rangle}}{\left(2\sinh\frac{mg_s}{2}\right)^2} . \tag{55}$$

Thus by summing up the KK-modes or (the number of $D0$ branes) we can avoid the integration of the proper time s which also removes the dependence on the cut-off. Thus we find the R_+^2 correction

$$\sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-m\langle q,t\rangle}}{\left(2\sinh \frac{mg_s}{2}\right)^2} \ . \tag{56}$$

as the free energy of topological string It is suggestive the partition function (the exponential of the free energy) has an infinite product form

$$Z = \prod_{n=1}^{\infty} \left(1 - q^n e^{-\beta(t)}\right)^{-n} \ , \tag{57}$$

where $q := e^{-g_s}$ and $\beta(t) := \langle q, t\rangle$.

It is easy to generalize the above computation to the case where the hypermultiplet has non-zero left spin $j_L = s$. The result is

$$Z = \prod_{n=1}^{\infty} \left(1 - q^{n+s} e^{-\beta(t)}\right)^{\pm n} \ , \tag{58}$$

where the sign is $+$ for fermions and $-$ for bosons. Summing up all the contributions, we see that the free energy of topological string tales the following form;

$$\log Z(t) = \sum_{j=0}^{\infty} \sum_{\beta \in H_2(X,\mathbb{Z})} n_{j,\beta} \left[\sum_{m=1}^{\infty} \left(2\sinh \frac{mg_s}{2}\right)^{2j-2} e^{-m\beta(t)} \right] \ . \tag{59}$$

In the trivial instanton sector $\beta = 0$, which is the contribution of constant maps to the topological string amplitude. The partition function

$$Z = \prod_{n} \frac{1}{(1-q^n)^n} = \sum_{\mathrm{P.P.}} q^{|\pi|} \tag{60}$$

coincides with the MacMahon function $M(q)$, which is known as the generating function of the plane partitions π (or $3D$ Young diagrams). In the strong coupling region $g_s \to \infty$, $|q| << 1$, we can expand the partition function

$$M(q) = 1 + q + 3\,q^2 + 6\,q^3 + 13\,q^4 + 24\,q^5 + 48\,q^6 + 86\,q^7 + 160\,q^8 + 282\,q^9 + 500\,q^{10} + \cdots \tag{61}$$

The coefficient of q^N gives the number of plane partitions ($3D$ Young tableaux) with $N = |\pi|$ boxes. This is what is the (quantum) crystal picture of topological string in the strong coupling, which also allows an interpretation in terms of $U(1)$ (topological) gauge theory on $D6$ brane.

To obtain an expansion of free energy in the weak coupling $g_s \to 0,\ q \to 1$, let us look at its Mellin transform with $t := g_s$

$$G(s) := \int_0^\infty \frac{dt}{t^{1-s}} F(t) = -\int_0^\infty \frac{dt}{t^{1-s}} \sum_{d=1}^\infty \sum_{n=1}^\infty \frac{n}{d} e^{-ndt} \ . \tag{62}$$

Exchanging the integral and sums, we obtain

$$G(s) = -\sum n^{1-s} d^{-1-s} \int_0^\infty \frac{dx}{x^{1-s}} e^{-x} = -\zeta(s-1)\zeta(s+1)\Gamma(s) \ . \tag{63}$$

The inversion formula of the Mellin transform is

$$F(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}\ (s)=s_0} ds\ G(s) t^{-s} \ , \tag{64}$$

where the contour is chosen to lie to the right of any pole of $G(s)$. Deforming the contour we see that the poles of $G(s)$ generate the laurent series expansion of $M(t)$. Now $\Gamma(s)$ has simple poles at $s = -n, n = 0, 1, 2, \cdots$ with residue $\frac{(-1)^n}{n!}$ and $\zeta(s)$ has simple pole at $s = 1$ with residue 1 and simple zeros at $s = -2, -4, -6, \cdots$. (All other zeros of $\zeta(s)$ are in the region $0 < \mathrm{Re}\ s < 1$ and they are actually on the line $\mathrm{Re}\ s = \frac{1}{2}$ is the famous Riemann conjecture.) Therefore $\zeta(s-1)\zeta(s+1)\Gamma(s)$ has simple pole at $s = 2$, double pole at $s = 0$ and simple poles at $s = -2n, n \in \mathbb{N}$. Note that the poles at $s = -1, -3, -5 \cdots$ are canceled by the zeros of the zeta function. Thus we obtain

$$\begin{aligned} F(t) =& -\left[\zeta(3)\Gamma(2)t^{-2} + \lim_{s\to 0}\frac{d}{ds}\left(\zeta(s-1)t^{-s}\right) + \sum_{n=1}^\infty \frac{1}{(2n)!}\zeta(1-2n)\zeta(-1-2n)t^{2n}\right] \\ =& \zeta(3)g_s^{-2} - \zeta'(-1) - \frac{1}{12}\log(-ig_s) + \sum_{g\geq 2} \frac{(-1)^g B_{2g} B_{2g-2}}{(2g-2)!2g(2g-2)} g_s^{2g-2} \ , \end{aligned} \tag{65}$$

where we have used $\zeta(-1) = -\frac{1}{12}, \zeta(1-2n) = -\frac{B_{2n}}{2n}$. The coefficients give the value of the Hodge integral on $\mathcal{M}_g$.

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